

Some of semileptonic and nonleptonic decays of B_c meson in a Bethe-Salpeter relativistic quark model

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The semileptonic decays $B_c^+ \rightarrow P(V) + \ell^+ + \bar{\nu}_\ell$ and the nonleptonic decays $B_c^+ \rightarrow P(V) + L$, where $P(V)$ denotes a pseudoscalar (vector) charmonium or ($\bar{b}s$)-meson, and L denotes a light meson, are studied in the framework of improved instantaneous Bethe-Salpeter (BS) equation and the Mandelstam formula. The numerical results (width and branching ratio of the decays) are presented in tables, and in order to compare conveniently, those obtained by other approaches are also put in the relevant tables. Based on the fact that the ratio $\frac{\mathcal{BR}(B_c^+ \rightarrow \psi(2S)\pi^+)}{\mathcal{BR}(B_c^+ \rightarrow J/\psi\pi^+)} = 0.24_{-0.040}^{+0.023}$ estimated here is in good agreement with the observation by the LHCb $\frac{\mathcal{BR}(B_c^+ \rightarrow \psi(2S)\pi^+)}{\mathcal{BR}(B_c^+ \rightarrow J/\psi\pi^+)} = 0.250 \pm 0.068(\text{stat}) \pm 0.014(\text{syst}) \pm 0.006(\mathcal{B})$, one may conclude that with respect to the decays the present framework works quite well.

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B_c meson carries two heavy flavor quantum numbers explicitly, and it decays only via weak interactions, although the strong and electromagnetic interactions can affect the decays. As consequences, B_c meson has a comparatively long lifetime and very rich weak decay channels with sizable branching ratios. Being an explicit double heavy flavor meson, its production cross section can be estimated by perturbative QCD quite reliably and one can conclude that only via strong interaction and at hadronic high energy collisions the meson can be produced so numerously that it can be observed experimentally [1–3]. Therefore, the meson is specially interesting in studying its production and decays both.

The first successful observation of B_c was achieved through the semileptonic decay channel $B_c \rightarrow J/\psi + \ell^+ + \bar{\nu}_\ell$ by CDF collaboration in 1998 from Run-I at Tevatron. They obtained the mass of B_c : $m_{B_c} = 6.40 \pm 0.39 \pm 0.13$ GeV and the lifetime: $\tau_{B_c} = 0.46_{-0.16}^{+0.18} \pm 0.03$ ps [4]. Later on CDF collaboration further gave a more precise mass $m_{B_c} = 6275.6 \pm 2.9(\text{stat}) \pm 5(\text{syst})$ MeV/ c^2 obtained through the exclusive non-leptonic decay $B_c \rightarrow J/\psi\pi^+$

[5], and upgraded their results [6]. In the meantime D0 collaboration at Tevatron also carried out the observations and confirmed CDF results [7]. Recently, LHCb reported several new observations on B_c decays [8]. Thus we may reasonably expect that at LHCb in the near future the B_c data will be largely enhanced and new results are issued in time.

In literatures, there are many works studying various B_c decays [9–28] under different approaches. Among the approaches in the market, the one used in Ref. [9] is that when the components in the concerned meson(s) in initial and final states are heavy quarks, an instantaneous Bethe-Salpeter (BS) equation [29] (also called Salpeter equation [30]) with an instantaneous QCD-inspired kernel (interaction)¹ is used to depict the meson(s) and the Mandelstam formula [31] is adopted to compute hadron matrix elements relevant to the concerned decays. This approach has a comparatively solid foundation because the relativistic ‘recoil effects’ in the decays² may be taken

¹ With the equation, the spectrum and relevant wave function as an eigenvalue problem derived from the BS equation can be computed.

² The difference between masses of the initial B_c meson and the decay product e.g. charmonium is great, so the recoil in a B_c decay must be relativistic, and the “recoil effects” in the decay should be taken into account well.

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into account better than in potential model and else approaches. The reason is that the BS equation and the Mandelstam formula both are established on relativistic quantum field theory, although the BS equation is deduced into an instantaneous one. Generally, when solving the instantaneous BS equation, the wave function needs to be formulated by a basis of angular momentum with the spin of its components according to the bound state quantum numbers, such as pseudoscalar or vector or else, whereas in Ref. [9] to do the formulation the authors, followed Ref. [30], took an extra approximation. Since now a way to solve the instantaneous BS equation without the extra approximation is available [32], and a way more properly to treat the relevant transition matrix elements in Mandelstam formulation has been explored for years [33], so we think that now it is the right time by using the new wave functions obtained by solving the instantaneous BS equation without the extra approximation and the improved formula for the transition matrix elements to estimate the B_c decays theoretically and then to compare the results with the newly experimental data to see how well the new improved approach [32, 33] works. Considering the progresses in experiments, especially those at LHCb, in this paper we would like to restrict ourselves to focus lights on the Cabibbo-Kobayashi-Maskawa (CKM) favored B_c decays: the semileptonic ones $B_c^+ \rightarrow P(V) + \ell^+ + \bar{\nu}_\ell$ and the nonleptonic ones $B_c^+ \rightarrow P(V) + \pi(\rho, K, K^*)$ precisely, where $P(V)$ represents pseudoscalar (vector) charmonium or a $\bar{b}s$ bound state.

The paper is organized as follows. In Sec. I we outline the useful formulas. In Sec. II we present numerical results for the semileptonic and nonleptonic decays and compare the results with those obtained by other approaches. Sec. III is contributed to discussions. We put the relativistic BS equation with covariant instantaneous approximation, the forms of relativistic wave functions for pseudoscalar and vector mesons, the formulations of the form factors, and the parameters used to solve the BS equation into Appendices.

I. FORMULATIONS FOR B_c SEMILEPTONIC AND NONLEPTONIC DECAYS

For the semileptonic decays $B_c^+ \rightarrow X + \ell^+ + \bar{\nu}_\ell$ shown in Fig. 1, the T -matrix element can be written as hadronic component and leptonic component:

$$T = \frac{G_F}{\sqrt{2}} V_{ij} \bar{u}_{\nu_\ell} \gamma^\mu (1 - \gamma_5) v_\ell \langle X(p', \epsilon) | J_\mu | B_c^+(p) \rangle, \quad (1)$$

where V_{ij} is the CKM matrix element, J_μ is the charged weak current responsible for the decays, p, p' are the momenta of the initial state B_c^+ and the final state X respectively, while ϵ is the polarization vector when X is a vector particle. The square of the matrix element,

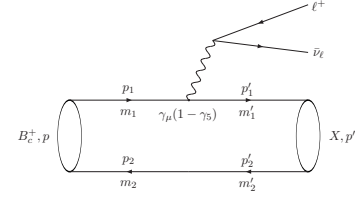


FIG. 1: Feynman diagram corresponding to the semileptonic decays $B_c^+ \rightarrow X + \ell^+ + \bar{\nu}_\ell$.

summed and averaged over the spin (unpolarized), is:

$$\sum |T|^2 = \frac{G_F^2}{2} |V_{ij}|^2 l^{\mu\nu} h_{\mu\nu}, \quad (2)$$

where the leptonic tensor:

$$l^{\mu\nu} \equiv \bar{u}_{\nu_\ell} \gamma^\mu (1 - \gamma_5) v_\ell \bar{u}_{\nu_\ell} (1 + \gamma_5) \gamma^\nu u_{\nu_\ell}, \quad (3)$$

is easy to compute, and the hadronic tensor is defined by:

$$h_{\mu\nu} \equiv \sum_\epsilon \langle B_c^+(p) | J_\nu^\dagger | X(p', \epsilon) \rangle \langle X(p', \epsilon) | J_\mu | B_c^+(p) \rangle. \quad (4)$$

where $J_\mu = V_\mu - A_\mu$. The general form of $h_{\mu\nu}$ based on Lorentz-covariance analysis can be written as:

$$\begin{aligned} h_{\mu\nu} = & -\alpha g_{\mu\nu} + \beta_{++} (p + p')_\mu (p + p')_\nu \\ & + \beta_{+-} (p + p')_\mu (p - p')_\nu + \beta_{-+} (p - p')_\mu (p + p')_\nu \\ & + \beta_{--} (p - p')_\mu (p - p')_\nu \\ & + i\gamma \epsilon_{\mu\nu\rho\sigma} (p + p')^\rho (p - p')^\sigma. \end{aligned} \quad (5)$$

By a straightforward calculation, the differential decay rate is obtained:

$$\begin{aligned} \frac{d^2\Gamma}{dx dy} = & |V_{ij}|^2 \frac{G_F^2 M^5}{32\pi^3} \left\{ \alpha \frac{(y - \frac{m_\ell^2}{M^2})}{M^2} + 2\beta_{++} \right. \\ & \times \left[2x(1 - \frac{M'^2}{M^2} + y) - 4x^2 - y + \frac{m_\ell^2}{4M^2} \right. \\ & \times \left. \left(8x + \frac{4M'^2 - m_\ell^2}{M^2} - 3y \right) \right. \\ & + 4(\beta_{+-} + \beta_{-+}) \frac{m_\ell^2}{M^2} (2 - 4x + y - \frac{2M'^2 - m_\ell^2}{M^2}) \\ & + 4\beta_{--} \frac{m_\ell^2}{M^2} (y - \frac{m_\ell^2}{M^2}) - \gamma \left[y(1 - \frac{M'^2}{M^2} - 4x + y) \right. \\ & \left. \left. + \frac{m_\ell^2}{M^2} (1 - \frac{M'^2}{M^2} + y) \right] \right\}, \end{aligned} \quad (6)$$

where $x \equiv E_\ell/M$ and $y \equiv (p - p')^2/M^2$, M is the mass of B_c^+ meson, M' is the mass of the final state X . The coefficient functions $\alpha, \beta_{++}, \beta_{+-}, \beta_{-+}, \beta_{--}$ and γ relate to the form factors of weak currents directly (see below).

To evaluate the exclusive semileptonic differential decay rates of B_c^+ meson, one needs to calculate the hadron

matrix element of the weak current J_μ sandwiched by the B_c^+ meson state as the initial state and a single-hadron state of the concerned final state, i.e., $\langle X(p', \epsilon) | J_\mu | B_c^+(p) \rangle$ with X being a given suitable meson. In fact, the hadron matrix elements of weak currents can be generally expressed in terms of the momenta p and p' of the mesons in initial state and final state respectively, as well as their coefficients. The coefficients, being functions of the momentum transfer $(p - p')$, are Lorentz-invariant and are called as form factors usually. As emphasised in Refs. [9, 13], with the help of the Mandelstam formalism [31], no matter how great the recoil happens the weak current hadron matrix element can be well calculated, thus here we adopt the method used in Refs. [9, 13] but with improvements [32, 33], i.e., the used wave functions are obtained by solving the relevant instantaneous BS equation with the improved approach [32, 33]. Although the improved approach is used to calculate the hadron matrix elements of weak currents, the form factors are still written as overlap integrals of the relevant wave functions for the bound states (mesons). To show the general feature of the improved approach, we put its outline in Appendices. Moreover, here we temperately constrain ourselves to consider the cases that X is a S -wave meson only.

According to the Mandelstam formalism and with wave functions of instantaneous BS equation(s), in the leading order, the matrix element $\langle X(p') | J_\mu | B_c^+(p) \rangle$ can be written as [33]:

$$\begin{aligned} & \langle X(p') | J_\mu | B_c^+(p) \rangle \\ &= \int \frac{d^4 q' d^4 q}{(2\pi)^4} Tr \left\{ \bar{\chi}_{p'} J_\mu \chi_p S_2^{(2)-1}(-p_2) \delta^4(p_2 - p'_2) \right\} \\ &= \int \frac{d^3 \vec{q}}{(2\pi)^3} Tr \left[\bar{\varphi}_{p'}^{++}(\vec{q} + \alpha'_2 \vec{r}) \gamma_\mu (1 - \gamma_5) \varphi_p^{++}(\vec{q}) \frac{\not{p}}{M} \right] \end{aligned} \quad (7)$$

here for the last equal sign we have chosen the center of mass system of initial meson B_c^+ ; $S_2(-p_2)$ is the propagator of the second component ("spectator"); \vec{r} is the three dimensional momentum of final hadron state X and $\alpha'_2 = m'_2/(m'_1 + m'_2)$; φ^{++} is the component of BS wave function projected onto the "positive energy" for the relevant mesons, and may be obtained by solving the BS equation. Its definition can be found in Appendix A. Since the initial and final states in the transition are both heavy mesons, as adopted in Eq. (7), it is a good approximation that only positive energy projected BS wave functions are included (the contributions from the component of the wave functions projected onto the "negative energy" are much smaller than that from the positive energy one).

The form factors can be generally related to the weak current matrix element as follows:

1. If X is a 1S_0 state, of the weak current the axial vector matrix element vanishes, and the vector current matrix element can be written as:

$$\langle X(p') | V_\mu | B_c^+(p) \rangle \equiv f_+(p + p')_\mu + f_-(p - p')_\mu. \quad (8)$$

2. If X is a 3S_1 state, of the weak current the axial

vector matrix element can be written as:

$$\begin{aligned} \langle X(p', \epsilon) | A_\mu | B_c^+(p) \rangle &\equiv f_\epsilon^* + a_+(\epsilon^* \cdot p)(p + p')_\mu \\ &\quad + a_-(\epsilon^* \cdot p)(p - p')_\mu, \end{aligned} \quad (9)$$

and the vector current matrix element as:

$$\langle X(p', \epsilon) | V_\mu | B_c^+(p) \rangle \equiv ig\epsilon_{\mu\nu\rho\sigma}\epsilon^{*\nu}(p + p')^\rho(p - p')^\sigma. \quad (10)$$

where ϵ is the polarization vector of the final hadron X .

With the relation between the matrix element and the form factors above and using Eq. (7), the form factors can be calculated out. Explicitly expressions for the form factors as overlap integrals of meson wave functions are given in Appendix B.3. Correspondingly, the coefficient functions α , β and γ in Eq. (6) can be expressed in terms of the form factors. For example, for the decay $B_c^+ \rightarrow P\ell^+\bar{\nu}_\ell$ (P is a pseudoscalar meson) we have:

$$\begin{aligned} \alpha &= \gamma = 0, \\ \beta_{++} &= f_+^2, \beta_{+-} = f_+f_-, \beta_{-+} = \beta_{+-}, \beta_{--} = f_-^2. \end{aligned} \quad (11)$$

For the decay $B_c^+ \rightarrow V\ell^+\bar{\nu}_\ell$ (V is a vector meson) we have:

$$\begin{aligned} \alpha &= f^2 + 4M^2\vec{p}^2g^2, \\ \beta_{++} &= \frac{f^2}{4M'^2} - M^2yg^2 + \frac{1}{2} \left[\frac{M^2}{M'^2}(1 - y) - 1 \right] fa_+ \\ &\quad + M^2 \frac{\vec{p}^2}{M'^2} a_+^2, \\ \beta_{+-} &= -\frac{f^2}{4M'^2} + (M^2 - M'^2)g^2 \\ &\quad + \frac{1}{4} \left[-\frac{M^2}{M'^2}(1 - y) - 3 \right] fa_+ \\ &\quad + \frac{1}{4} \left[\frac{M^2}{M'^2}(1 - y) - 1 \right] fa_- + M^2 \frac{\vec{p}^2}{M'^2} a_+a_-, \\ \beta_{-+} &= \beta_{+-}, \\ \beta_{--} &= \frac{f^2}{4M'^2} + [M^2y - 2(M^2 + M'^2)]g^2 \\ &\quad + \frac{1}{2} \left[-\frac{M^2}{M'^2}(1 - y) - 3 \right] fa_- + M^2 \frac{\vec{p}^2}{M'^2} a_-^2, \\ \gamma &= 2fg. \end{aligned} \quad (12)$$

Putting the above form factors into the formula for differential decay rates Eq. (6), the concerned semileptonic decay rates can be calculated.

For the nonleptonic decays $B_c^+ \rightarrow X + \pi(K, \rho, K^*)$ concerned here, we follow Ref. [9] to take the CKM-favored effective Hamiltonian with QCD leading logarithm correction to be responsible for them:

$$\begin{aligned} H_{eff}^b &= \frac{G_F}{\sqrt{2}} V_{cb} [c_1^b(\mu_b) Q_1^{cb} + c_2^b(\mu_b) Q_2^{cb}] + h.c., \\ H_{eff}^c &= \frac{G_F}{\sqrt{2}} V_{cs} [c_1^c(\mu_c) Q_1^{cs} + c_2^c(\mu_c) Q_2^{cs}] + h.c. \end{aligned} \quad (13)$$

where $c_i^c(\mu_c) = c_i^c(m_c)$ and $c_i^b(\mu_b) = c_i^c(m_b)$ are the Wilson coefficients, and the four-fermion operators Q_1^{ij} and Q_2^{ij} are defined:

$$\begin{aligned} Q_1^{bc} &\equiv [(\bar{d}'u)_{V-A} + (\bar{s}'c)_{V-A}](\bar{c}b)_{V-A}, \\ Q_2^{bc} &\equiv (\bar{c}c)_{V-A}(\bar{s}'b)_{V-A} + (\bar{c}u)_{V-A}(\bar{d}'b)_{V-A}, \\ Q_1^{cs} &\equiv (\bar{c}s)_{V-A}(\bar{d}'u)_{V-A}, \\ Q_2^{cs} &\equiv (\bar{d}'s)_{V-A}(\bar{c}u)_{V-A}, \end{aligned}$$

d' and s' denote 'down' and 'strange' weak eigenstates³. Based on the QCD Renormalization Group (RG) calculation, and in terms of the combination operators $Q_{\pm} = (Q_1 \pm Q_2)$ which have diagonal anomalous dimensions, the corresponding Wilson coefficients read as follows [9, 34]:

$$\begin{aligned} c_+^c(\mu) &= \left[\frac{\alpha_s(M_W)}{\alpha_s(m_b)} \right]^{6/23} \left[\frac{\alpha_s(m_b)}{\alpha_s(\mu)} \right]^{6/25}, \\ c_-^c(\mu) &= [c_+^c(\mu)]^{-2}, \\ c_+^b(\mu) &= \left[\frac{\alpha_s(M_W)}{\alpha_s(m_b)} \right]^{6/23} \left[\frac{\alpha_s(m_b)}{\alpha_s(\mu)} \right]^{-3/25}, \\ c_-^b(\mu) &= \left[\frac{\alpha_s(M_W)}{\alpha_s(m_b)} \right]^{-12/23} \left[\frac{\alpha_s(m_b)}{\alpha_s(\mu)} \right]^{-12/25}, \end{aligned} \quad (14)$$

Then to use "naive factorization" as done in Ref. [9], the T -matrix element can be written as:

$$T = \frac{G_F}{\sqrt{2}} V_{ij} V_{lk} a_1 \langle L(k, \epsilon') | J^\mu | 0 \rangle \langle X(p', \epsilon) | J_\mu | B_c^+(p) \rangle, \quad (15)$$

where V_{ij}, V_{lk} are the relevant CKM matrix elements to L and X accordingly, $L = \pi(K, \rho, K^*)$, p, p', k are the momenta of B_c , X and $\pi(K, \rho, K^*)$ respectively, and ϵ', ϵ are the polarization vectors for ρ or K^* and X when X is a vector meson. The parameter

$$a_1 = c_1(\mu) + \xi c_2(\mu), \quad \xi = \frac{1}{N_c} \quad (16)$$

in Eq. (15) is attributed to the contribution from the operators Q_1 and that from the Fierz-reordered Q_2 with a suppressed factor ξ to the concerned decays [9].

For the two-body decays $B_c^+ \rightarrow X + L^+$ concerned here, having the T matrix element Eq. (15), it is straightforward to calculate the decay widths.

II. NUMERICAL RESULTS

The components of the meson B_c are \bar{b} and c quarks, and it happens that the contributions from each of them

to the total decay rate are comparable in magnitude. Thus the semileptonic decay modes of B_c meson can be classified into two: \bar{b} -quark decays with the c quark inside the meson as a spectator, and c -quark decays with the \bar{b} quark as a spectator. The former causes B_c decays into charmonium or D -meson pair, while the latter causes B_c decays into B_s or B mesons. In this paper, we restrict ourselves to compute B_c decays to charmonium or B_s meson only because the approach adopted here is good for double heavy mesons.

When calculating the decays under the adopted approach, we need to fix several parameters. In fact, the parameters are fixed by fitting well-measured experimental data and the established potential model. The parameters appearing in the potential (the kernel of Salpeter equation) used in this work are fixed by the spectra of heavy quarkonia as done in Ref. [33] and outlined in Appendix B.4. The masses of the ground states are used as inputs, while the masses of excited states are considered as predictions. According to the fits we obtain $M_{\eta_c(2S)} = 3.576$ GeV and $M_{\psi(2S)} = 3.686$ GeV, and to compare with experimental data $M_{\eta_c'}^{exp} = 3.637$ GeV and $M_{\psi'}^{exp} = 3.686$ GeV, one may see the fits are quite good.

The values of the CKM matrix elements adopted in this paper are $V_{cb} = 0.0406$, $V_{cs} = 0.9735$, $V_{ud} = 0.974$ and $V_{us} = 0.2252$. The properties of relevant light mesons appearing in the concerned nonleptonic decays are served as phenomenological inputs, namely we take

$$\begin{aligned} M_\pi &= 0.140 \text{ GeV}, & f_\pi &= 0.130 \text{ GeV}, \\ M_\rho &= 0.775 \text{ GeV}, & f_\rho &= 0.205 \pm 0.009 \text{ GeV}, \\ M_K &= 0.494 \text{ GeV}, & f_K &= 0.156 \text{ GeV}, \\ M_{K^*} &= 0.892 \text{ GeV}, & f_{K^*} &= 0.217 \pm 0.005 \text{ GeV}, \end{aligned}$$

where the masses and the decay constants are taken from PDG [35], except f_ρ and f_{K^*} , which are quoted from Ref. [36].

The numerical results of semileptonic decays are presented in Table I, and here the uncertainties for our results are obtained by varying the model parameters m_b , m_c , m_s , λ and Λ_{QCD} by $\pm 5\%$. For comparison precisely, the results from other typical approaches are also listed in the tables. To see the feature of the decays, we plot the lepton spectrum for the decays $B_c^+ \rightarrow P(V) + \ell^+ + \bar{\nu}_\ell$ in Fig. 2 and Fig. 3 respectively.

The concerned B_c nonleptonic decay modes (some for \bar{b} -decays and c as spectator and some for c -decays and \bar{b} as a spectator) are computed with uncertainties precisely too. The results, as well as some from other approaches for comparisons, are presented in Table II and Table III, respectively.

III. DISCUSSION AND CONCLUSION

If comparing the semileptonic and nonleptonic decays estimated by various approaches via Tables I-III, one may find that the deviations among the theoretical predictions

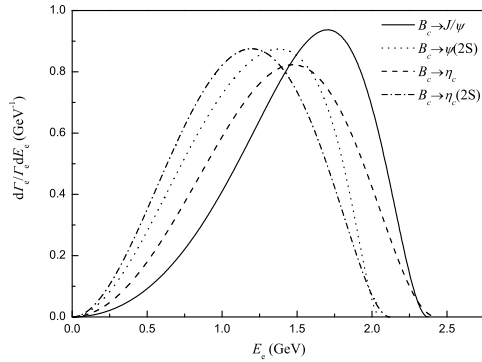
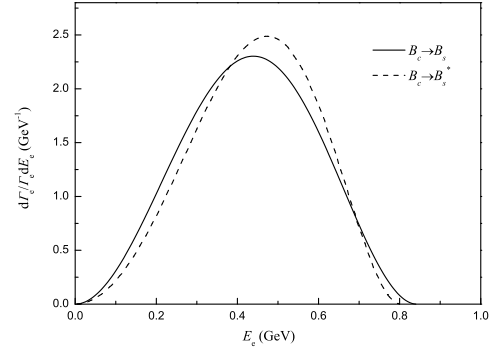
³ Since we restrict ourselves to consider the decays $B_c^+ \rightarrow X + \pi(K, \rho, K^*)$ here, so we list the main operators the Q_1^{ij} and Q_2^{ij} only which relate and greatly contribute to the decays.

TABLE I: The decay widths of the exclusive semileptonic decay modes (in 10^{-15} GeV).

Mode	Ours	[9]	[11]	[15]	[16]	[17]	[18]	[19]	[21]	[22]	[23]	[24]
$B_c^+ \rightarrow \eta_c e^+ \bar{\nu}_e$	$8.02^{+0.36}_{-0.39}$	14.2	11	11.1	13.05	5.9	14	10	4.3	10.6	8.31	6.5
$B_c^+ \rightarrow B_s e^+ \bar{\nu}_e$	$19.7^{+2.0}_{-2.1}$	26.6	59	14.3	22.0	12	29	18	11.75	16.4	26.8	11.1
$B_c^+ \rightarrow J/\psi e^+ \bar{\nu}_e$	$25.2^{+0.7}_{-0.8}$	34.4	28	30.2	26.6	17.7	33	42	16.8	38.5	20.3	21.8
$B_c^+ \rightarrow B_s^* e^+ \bar{\nu}_e$	$39.9^{+0.6}_{-1.3}$	44.0	65	50.4	51.2	25	37	43	32.56	40.9	34.6	43.7
$B_c^+ \rightarrow \eta_c(2S) e^+ \bar{\nu}_e$	$0.969^{+0.075}_{-0.088}$	0.727	0.28			0.46					0.605	
$B_c^+ \rightarrow \psi(2S) e^+ \bar{\nu}_e$	$1.49^{+0.20}_{-0.25}$	1.45	1.36			0.44					0.186	

TABLE II: The decay widths of the exclusive nonleptonic decay modes with c -quark spectator (in 10^{-15} GeV).

Mode	Ours	[9]	[11]	[15]	[16]	[17]	[22]	[23]
$J/\psi + \pi$	$1.24^{+0.11}_{-0.11} a_1^2$	$1.97 a_1^2$	$1.43 a_1^2$	$1.22 a_1^2$	$0.82 a_1^2$	$0.67 a_1^2$	$1.79 a_1^2$	$1.01 a_1^2$
$J/\psi + K$	$0.0949^{+0.0080}_{-0.0081} a_1^2$	$0.152 a_1^2$	$0.12 a_1^2$	$0.090 a_1^2$	$0.079 a_1^2$	$0.052 a_1^2$	$0.130 a_1^2$	$0.0764 a_1^2$
$J/\psi + \rho$	$3.59^{+0.64}_{-0.58} a_1^2$	$5.95 a_1^2$	$4.37 a_1^2$	$3.48 a_1^2$	$2.32 a_1^2$	$1.8 a_1^2$	$5.07 a_1^2$	$3.25 a_1^2$
$J/\psi + K^*$	$0.226^{+0.028}_{-0.029} a_1^2$	$0.324 a_1^2$	$0.25 a_1^2$	$0.197 a_1^2$	$0.18 a_1^2$	$0.11 a_1^2$	$0.263 a_1^2$	$0.174 a_1^2$
$\psi(2S) + \pi$	$0.298^{+0.002}_{-0.002} a_1^2$	$0.251 a_1^2$				$0.12 a_1^2$		$0.0708 a_1^2$
$\psi(2S) + K$	$0.0218^{+0.0003}_{-0.0017} a_1^2$	$0.018 a_1^2$				$0.009 a_1^2$		$0.00499 a_1^2$
$\psi(2S) + \rho$	$0.765^{+0.090}_{-0.123} a_1^2$	$0.710 a_1^2$				$0.20 a_1^2$		$0.183 a_1^2$
$\psi(2S) + K^*$	$0.0459^{+0.0037}_{-0.0059} a_1^2$	$0.038 a_1^2$				$0.011 a_1^2$		$0.00909 a_1^2$
$\eta_c + \pi$	$1.18^{+0.10}_{-0.10} a_1^2$	$2.07 a_1^2$	$1.8 a_1^2$	$1.59 a_1^2$	$1.47 a_1^2$	$0.93 a_1^2$	$1.71 a_1^2$	$1.49 a_1^2$
$\eta_c + K$	$0.0919^{+0.0078}_{-0.0078} a_1^2$	$0.161 a_1^2$	$0.15 a_1^2$	$0.119 a_1^2$	$0.15 a_1^2$	$0.073 a_1^2$	$0.127 a_1^2$	$0.115 a_1^2$
$\eta_c + \rho$	$2.89^{+0.51}_{-0.46} a_1^2$	$5.48 a_1^2$	$4.5 a_1^2$	$3.74 a_1^2$	$3.35 a_1^2$	$2.3 a_1^2$	$4.04 a_1^2$	$3.93 a_1^2$
$\eta_c + K^*$	$0.172^{+0.022}_{-0.021} a_1^2$	$0.286 a_1^2$	$0.22 a_1^2$	$0.200 a_1^2$	$0.24 a_1^2$	$0.12 a_1^2$	$0.203 a_1^2$	$0.198 a_1^2$
$\eta_c(2S) + \pi$	$0.322^{+0.010}_{-0.014} a_1^2$	$0.268 a_1^2$				$0.19 a_1^2$		$0.248 a_1^2$
$\eta_c(2S) + K$	$0.0242^{+0.0008}_{-0.0012} a_1^2$	$0.020 a_1^2$				$0.014 a_1^2$		$0.0184 a_1^2$
$\eta_c(2S) + \rho$	$0.711^{+0.094}_{-0.095} a_1^2$	$0.622 a_1^2$				$0.40 a_1^2$		$0.587 a_1^2$
$\eta_c(2S) + K^*$	$0.0408^{+0.0037}_{-0.0040} a_1^2$	$0.031 a_1^2$				$0.021 a_1^2$		$0.0283 a_1^2$

FIG. 2: The lepton energy spectrum for the semileptonic decays $B_c^+ \rightarrow \eta_c(\eta_c(2S), J/\psi, \psi(2S))\ell^+ \bar{\nu}_\ell$.FIG. 3: The lepton energy spectrum for the semileptonic decays $B_c^+ \rightarrow B_s(B_s^*)\ell^+ \bar{\nu}_\ell$.

by the various approaches are quite wide. Specifically, the results with new solutions of the Salpeter equation and new formulation are quite different from those in Ref. [9] too.

When calculating the decay branching ratio of semileptonic and nonleptonic decays, here the lifetime of B_c meson is needed as input. For this purpose, we take the experimental lifetime from PDG [35]. For the nonleptonic decays considered here, the parameter a_1 for nonleptonic decays appearing in Eq. (15), additionally, need to eval-

uate precisely too. Note that a_1 for b quark (denoted as a_1^b) decays should be different from a_1 for c quark (denoted as a_1^c) decays, and we take $a_1^b = 1.14$ and $a_1^c = 1.2$ as in Refs. [11, 17, 18, 25, 27]. Having the lifetime and the parameter a_1 fixed, the branching ratio of the concerned decay modes are straightforwardly calculated and we put the results in Table IV and Table V respectively.

Recently, LHCb has reported an observation of decays

TABLE III: The decay widths of the exclusive nonleptonic decay modes with b -quark spectator (in 10^{-15} GeV).

Mode	Ours	[9]	[11]	[15]	[16]	[17]	[22]	[23]
$B_s + \pi$	$46.5_{-5.9}^{+6.2} a_1^2$	$58.4 a_1^2$	$167 a_1^2$	$15.8 a_1^2$	$34.8 a_1^2$	$25 a_1^2$	$44.0 a_1^2$	$65.1 a_1^2$
$B_s + K$	$3.55_{-0.37}^{+0.38} a_1^2$	$4.20 a_1^2$	$10.7 a_1^2$	$1.70 a_1^2$		$2.1 a_1^2$	$3.28 a_1^2$	$4.69 a_1^2$
$B_s + \rho$	$26.5_{-3.9}^{+4.2} a_1^2$	$44.8 a_1^2$	$72.5 a_1^2$	$39.2 a_1^2$	$23.6 a_1^2$	$14 a_1^2$	$20.2 a_1^2$	$42.7 a_1^2$
$B_s + K^*$	$0.0862_{-0.0075}^{+0.0078} a_1^2$			$1.06 a_1^2$		$0.03 a_1^2$		$0.292 a_1^2$
$B_s^* + \pi$	$31.4_{-2.4}^{+2.0} a_1^2$	$51.6 a_1^2$	$66.3 a_1^2$	$12.5 a_1^2$	$19.8 a_1^2$	$16 a_1^2$	$34.7 a_1^2$	$25.3 a_1^2$
$B_s^* + K$	$1.66_{-0.09}^{+0.06} a_1^2$	$2.96 a_1^2$	$3.8 a_1^2$	$1.34 a_1^2$		$1.1 a_1^2$	$2.52 a_1^2$	$1.34 a_1^2$
$B_s^* + \rho$	$139_{-12}^{+11} a_1^2$	$150 a_1^2$	$204 a_1^2$	$171 a_1^2$	$123 a_1^2$	$110 a_1^2$	$152.1 a_1^2$	$139.6 a_1^2$

TABLE IV: The branching ratio (in %) of the exclusive semileptonic decay modes with the lifetime of the B_c : $\tau_{B_c} = 0.452 ps$.

Mode	BR (%)
$B_c^+ \rightarrow \eta_c e^+ \bar{\nu}_e$	$0.551_{-0.027}^{+0.025}$
$B_c^+ \rightarrow B_s e^+ \bar{\nu}_e$	$1.35_{-0.14}^{+0.14}$
$B_c^+ \rightarrow J/\psi e^+ \bar{\nu}_e$	$1.73_{-0.05}^{+0.05}$
$B_c^+ \rightarrow B_s^* e^+ \bar{\nu}_e$	$2.74_{-0.09}^{+0.04}$
$B_c^+ \rightarrow \eta_c(2S) e^+ \bar{\nu}_e$	$0.0665_{-0.0060}^{+0.0052}$
$B_c^+ \rightarrow \psi(2S) e^+ \bar{\nu}_e$	$0.103_{-0.018}^{+0.013}$

$B_c^+ \rightarrow \psi \pi^+$ and $B_c^+ \rightarrow \psi(2S) \pi^+$ i.e. the related ratio [8]

$$\frac{\mathcal{BR}(B_c^+ \rightarrow \psi(2S) \pi^+)}{\mathcal{BR}(B_c^+ \rightarrow J/\psi \pi^+)} = 0.250 \pm 0.068(\text{stat}) \pm 0.014(\text{syst}) \pm 0.006(\mathcal{B}). \quad (17)$$

We would like to point out that, in contrary to the others observables, the above measured ratio, in which the production of B_c meson is canceled totally, is a very essential test of the decays thus here we precisely give the corresponding ratio given by the approach adopted here:

$$\frac{\mathcal{BR}(B_c^+ \rightarrow \psi(2S) \pi^+)}{\mathcal{BR}(B_c^+ \rightarrow J/\psi \pi^+)} = 0.24_{-0.040}^{+0.023}, \quad (18)$$

and one may see that it is in good agreement with the observation. Here we should further note that the parameter a_1 which appears in Eq. (15) and the theoretical uncertainties caused by naive factorization for the nonleptonic decays would be canceled a lot in calculating the ratio. Namely the related ratio is mostly determined by hadron transition, so this agreement between the experimental value and the theoretical estimate on the ratio indicates a vary strong support of the present approach.

In summary, we have calculated the decay width and branching ratio of the exclusive semileptonic decays of B_c meson to a charmonium or a B_s meson plus leptons and nonleptonic decays to a charmonium or a B_s meson plus a light meson under the improved instantaneous BS equation and Mandelstam approach. Under this approach, the full Salpeter equations for $(\bar{b}c)$, $(\bar{b}s)$ and $(\bar{c}c)$ etc systems are solved with the respective full relativistic wave functions for $J^P = 0^-$ and $J^P = 1^-$ states. To calculate

TABLE V: The branching ratio (in %) of the exclusive nonleptonic decay modes with the lifetime of the B_c : $\tau_{B_c} = 0.452 ps$.

Mode	BR (%)	Mode	BR (%)
$J/\psi + \pi$	$0.111_{-0.010}^{+0.009}$	$\eta_c + \pi$	$0.105_{-0.009}^{+0.009}$
$J/\psi + K$	$8.47_{-0.73}^{+0.71} \times 10^{-3}$	$\eta_c + K$	$8.21_{-0.70}^{+0.69} \times 10^{-3}$
$J/\psi + \rho$	$0.320_{-0.051}^{+0.058}$	$\eta_c + \rho$	$0.258_{-0.041}^{+0.046}$
$J/\psi + K^*$	$0.0201_{-0.0025}^{+0.0026}$	$\eta_c + K^*$	$0.0154_{-0.0019}^{+0.0019}$
$\psi(2S) + \pi$	$0.0266_{-0.0020}^{+0.0002}$	$\eta_c(2S) + \pi$	$0.0287_{-0.0012}^{+0.0009}$
$\psi(2S) + K$	$1.94_{-0.15}^{+0.03} \times 10^{-3}$	$\eta_c(2S) + K$	$2.16_{-0.10}^{+0.07} \times 10^{-3}$
$\psi(2S) + \rho$	$0.0683_{-0.0110}^{+0.0080}$	$\eta_c(2S) + \rho$	$0.0634_{-0.0084}^{+0.0085}$
$\psi(2S) + K^*$	$4.10_{-0.53}^{+0.32} \times 10^{-3}$	$\eta_c(2S) + K^*$	$3.64_{-0.36}^{+0.34} \times 10^{-3}$
$B_s + \pi$	$4.60_{-0.59}^{+0.61}$	$B_s^* + \pi$	$3.11_{-0.24}^{+0.19}$
$B_s + K$	$0.352_{-0.037}^{+0.036}$	$B_s^* + K$	$0.164_{-0.009}^{+0.006}$
$B_s + \rho$	$2.62_{-0.39}^{+0.42}$	$B_s^* + \rho$	$13.8_{-1.2}^{+1.0}$
$B_s + K^*$	$8.53_{-0.74}^{+0.77} \times 10^{-3}$		

the hadron transition matrix elements, the Mandelstam formula has been used and it is suitably approximated to fit the instantaneous approximation. We find that the results with this approach seem in certain degree to have been improved in comparison with those obtained by the early ones in Ref. [9] with the more approximated formulation. We also should point out that since only the experimental related ratio $\frac{\mathcal{BR}(B_c^+ \rightarrow \psi(2S) \pi^+)}{\mathcal{BR}(B_c^+ \rightarrow J/\psi \pi^+)}$ is available now, and the two involved decay modes in the ratio are two-body decays, so the test of the approaches are limited. Thus we think that to conclude about all the approaches in literature, more experimental data of the semileptonic decays, e.g. the decay spectrum of the positron, and more related ratios of various nonleptonic decays which are independent on the production of B_c meson etc are requested.

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Appendix A: INSTANTANEOUS BS EQUATION

BS equation for a quark-antiquark bound state generally is written as:

$$\chi_P(q) = \frac{1}{\not{p}_1 - m_1} i \int \frac{d^4 k}{(2\pi)^4} V(P, k, q) \chi_P(k) \frac{1}{\not{p}_2 + m_2} \quad (\text{A1})$$

where $p_1, p_2; m_1, m_2$ are the momenta and masses of the quark and anti-quark, respectively. $\chi_P(q)$ is the BS wave function with the total momentum P and relative momentum q , $V(P, k, q)$ is the kernel between the quark-antiquark in the bound state. P and q are defined as:

$$p_1 = \alpha_1 P + q, \quad \alpha_1 = \frac{m_1}{m_1 + m_2},$$

$$p_2 = \alpha_2 P - q, \quad \alpha_2 = \frac{m_2}{m_1 + m_2}.$$

Moreover, the BS wave function $\chi_P(q)$ satisfies the normalization condition:

$$\int \frac{d^4 k d^4 q}{(2\pi)^4} \text{Tr} \left\{ \bar{\chi}_P(k) \frac{\partial}{\partial P_0} [S_1^{-1}(p_1) S_2^{-1}(p_2) \delta^4(k - q) + V(P, k, q)] \chi_P(q) \right\} = 2iP_0, \quad (\text{A2})$$

where $S_1(p_1)$ and $S_2(p_2)$ are the propagators of the quark and anti-quark, respectively.

In general, the BS equation in four dimensional ‘relative’ space-time is hard to solve comparatively. Whereas if the bound states are formed by heavy components (quarks) then the kernel of the equation may approximately become an instantaneous one, and one may overcome the difficulty to solve the equation in four dimensional ‘relative’ space-time instead by adopting a so-called instantaneous approximate approach to turn the equation into a one in three ‘relative’ space. The proposal by Salpeter [30] is the approach, that the time-like component of the relative momentum is integrated out in terms of a contour integration so the BS equation in four dimension is reduced to a one in three dimension finally when the kernel is an instantaneous one. For double heavy bound states, here we follow the Salpeter approach but less approximations than he did. Let us outline our approach (Salpeter’s with less approximations) here. The approximately instantaneous kernel has the following form:

$$V(P, k, q) \sim V(|\mathbf{k} - \mathbf{q}|), \quad (\text{A3})$$

especially, it is the case, when the two constituents of meson is very heavy.

Since the recoil in momentum may be great for the concerned B_c semileptonic decays, so for convenience even under instantaneous approximation we reduce and solve the BS equation in a Lorentz covariant form, i.e., to divide the relative momentum q into two parts, $q_{P\parallel}$ and $q_{P\perp}$, a parallel part and an orthogonal one to P , respectively:

$$q^\mu = q_{P\parallel}^\mu + q_{P\perp}^\mu, \quad (\text{A4})$$

where $q_{P\parallel}^\mu \equiv (P \cdot q / M^2) P^\mu$, $q_{P\perp}^\mu \equiv q^\mu - q_{P\parallel}^\mu$, and M is the mass of the relevant meson. Correspondingly, we have two Lorentz invariant variables:

$$q_P = \frac{P \cdot q}{M}, \quad q_{P_T} = \sqrt{q_{P\perp}^2 - q^2} = \sqrt{-q_{P\perp}^2}. \quad (\text{A5})$$

It is easy to see that they turn to the usual component q_0 and $|\vec{q}|$ if in the frame of $\vec{P} = 0$. In the same sense, the volume element of a relative momentum k can be written in an invariant form:

$$d^4 k = dk_P k_{P_T}^2 dk_{P_T} ds d\phi, \quad (\text{A6})$$

where ϕ is the azimuthal angle, $s = (k_P q_P - k \cdot q) / (k_{P_T} q_{P_T})$. So now the instantaneous interaction kernel Eq. (A3) can be rewritten as:

$$V(|\vec{k} - \vec{q}|) = V(k_{P\perp}, s, q_{P\perp}). \quad (\text{A7})$$

If we introduce two notations as below:

$$\eta(q_{P\perp}^\mu) \equiv \int \frac{k_{P_T}^2 dk_{P_T} ds}{(2\pi)^2} V(k_{P\perp}, s, q_{P\perp}) \varphi_P(k_{P\perp}^\mu),$$

$$\varphi_P(q_{P\perp}^\mu) \equiv i \int \frac{dq_P}{2\pi} \chi_P(q_{P\parallel}^\mu, q_{P\perp}^\mu). \quad (\text{A8})$$

Then the BS equation can be take the form as follows:

$$\chi_P(q_{P\parallel}^\mu, q_{P\perp}^\mu) = S_1(p_1^\mu) \eta(q_{P\perp}^\mu) S_2(p_2^\mu). \quad (\text{A9})$$

The propagator of the relevant particles with masses m_1 and m_2 can be decomposed as:

$$S_i(p_i^\mu) = \frac{\Lambda_{i_P}^+(q_{P\perp}^\mu)}{J(i)q_P + \alpha_i M - \omega_{i_P} + i\varepsilon} + \frac{\Lambda_{i_P}^-(q_{P\perp}^\mu)}{J(i)q_P + \alpha_i M + \omega_{i_P} - i\varepsilon}, \quad (\text{A10})$$

with

$$\omega_{i_P} = \sqrt{m_i^2 + q_{P_T}^2},$$

$$\Lambda_{i_P}^\pm(q_{P\perp}^\mu) = \frac{1}{2\omega_{i_P}} \left[\frac{P}{M} \omega_{i_P} \pm J(i)(m_i + \not{q}_{P\perp}) \right] \quad (\text{A11})$$

where $i=1, 2$ for the quark and anti-quark, respectively, and $J(i) = (-1)^{i+1}$. $\Lambda_{i_P}^\pm(q_{P\perp}^\mu)$ satisfies the relations as follows:

$$\Lambda_{i_P}^+(q_{P\perp}^\mu) + \Lambda_{i_P}^-(q_{P\perp}^\mu) = \frac{P}{M},$$

$$\Lambda_{i_P}^\pm(q_{P\perp}^\mu) \frac{P}{M} \Lambda_{i_P}^\pm(q_{P\perp}^\mu) = \Lambda_{i_P}^\pm(q_{P\perp}^\mu),$$

$$\Lambda_{i_P}^\pm(q_{P\perp}^\mu) \frac{P}{M} \Lambda_{i_P}^\mp(q_{P\perp}^\mu) = 0. \quad (\text{A12})$$

In fact, Λ^\pm may be considered as ‘covariant energy-projection’ operators, i.e., in the rest frame $\vec{P} = 0$, they turn to the energy projection operator.

Introducing notations:

$$\varphi_P^{\pm\pm}(q_{P\perp}^\mu) \equiv \Lambda_{1P}^\pm(q_{P\perp}^\mu) \frac{P}{M} \varphi_P(q_{P\perp}^\mu) \frac{P}{M} \Lambda_{2P}^\pm(q_{P\perp}^\mu), \quad (\text{A13})$$

and taking into account $\frac{E}{M} \frac{E}{M} = 1$, we have:

$$\begin{aligned} \varphi_P(q_{P\perp}^\mu) &= \varphi_P^{++}(q_{P\perp}^\mu) + \varphi_P^{+-}(q_{P\perp}^\mu) \\ &\quad + \varphi_P^{-+}(q_{P\perp}^\mu) + \varphi_P^{--}(q_{P\perp}^\mu). \end{aligned}$$

Let us further integrate q_P out on both sides of Eq. (A9), and obtain:

$$\begin{aligned} \varphi_P(q_{P\perp}^\mu) &= \frac{\Lambda_{1P}^+(q_{P\perp}^\mu) \eta_P(q_{P\perp}^\mu) \Lambda_{2P}^+(q_{P\perp}^\mu)}{M - \omega_{1P} - \omega_{2P}} \\ &\quad - \frac{\Lambda_{1P}^-(q_{P\perp}^\mu) \eta_P(q_{P\perp}^\mu) \Lambda_{2P}^-(q_{P\perp}^\mu)}{M + \omega_{1P} + \omega_{2P}}. \end{aligned}$$

We decompose it into the coupled equations:

$$\begin{aligned} (M - \omega_{1P} - \omega_{2P}) \varphi_P^{++}(q_{P\perp}^\mu) &= \Lambda_{1P}^+(q_{P\perp}^\mu) \eta_P(q_{P\perp}^\mu) \Lambda_{2P}^+(q_{P\perp}^\mu), \\ (M + \omega_{1P} + \omega_{2P}) \varphi_P^{--}(q_{P\perp}^\mu) &= -\Lambda_{1P}^-(q_{P\perp}^\mu) \eta_P(q_{P\perp}^\mu) \Lambda_{2P}^-(q_{P\perp}^\mu), \\ \varphi_P^{+-}(q_{P\perp}^\mu) &= \varphi_P^{-+}(q_{P\perp}^\mu) = 0. \end{aligned} \quad (\text{A14})$$

Correspondingly, the normalization condition of Eq. (A2) in covariant form reads:

$$\int \frac{q_{P_T}^2 dq_{P_T}}{(2\pi)^2} \text{tr} \left[\bar{\varphi}^{++} \frac{P}{M} \varphi^{++} \frac{P}{M} - \bar{\varphi}^{--} \frac{P}{M} \varphi^{--} \frac{P}{M} \right] = 2P_0.$$

If binding is weak, the positive energy components of the wave functions φ^{++} are large owing to having a very small factor $(M - \omega_{1P} - \omega_{2P})$, so one can keep the first equation of Eq. (A14) only, and safely dropped the rest equations at the lowest-order approximation. In Ref. [9] it is the case for the heavy quarkonium and B_c meson.

Appendix B: Precise equation and weak current matrix elements

The wave functions appearing in the Mandelstam formulas for transition matrix elements are the solution of the corresponding BS equation, so let us show here how to obtain the "precise equation" (all the equations for $\varphi^{\pm\pm}$ are taken into account) and to solve the equation for a concerned heavy meson.

1. Equation and solution for heavy pseudoscalar meson

The relativistic wave function for heavy pseudoscalar mesons with the quantum numbers $J^P = 0^-$ can be generally written as the four terms constructed by P , $q_{P\perp}$ and gamma matrices [37]:

$$\begin{aligned} \varphi_{0-}(q_{P\perp}) &= \left[f_1(q_{P\perp}) P + f_2(q_{P\perp}) M + f_3(q_{P\perp}) \not{q}_{P\perp} \right. \\ &\quad \left. + f_4(q_{P\perp}) \frac{P \not{q}_{P\perp}}{M} \right] \gamma_5, \end{aligned} \quad (\text{B1})$$

where M is the mass of the pseudoscalar meson. Due to the last two equations of Eq. (A14): $\varphi_{0-}^{+-} = \varphi_{0-}^{-+} = 0$, we have:

$$\begin{aligned} f_3(q_{P\perp}) &= \frac{f_2(q_{P\perp}) M (-\omega_1 + \omega_2)}{m_2 \omega_1 + m_1 \omega_2}, \\ f_4(q_{P\perp}) &= -\frac{f_1(q_{P\perp}) M (\omega_1 + \omega_2)}{m_2 \omega_1 + m_1 \omega_2}. \end{aligned} \quad (\text{B2})$$

Then there are only two independent wave functions $f_1(q_{P\perp})$ and $f_2(q_{P\perp})$ being left in the Eq. (B1):

$$\begin{aligned} \varphi_{0-}(q_{P\perp}) &= \left[f_1(q_{P\perp}) P + f_2(q_{P\perp}) M \right. \\ &\quad \left. - f_2(q_{P\perp}) \not{q}_{P\perp} \frac{M(\omega_1 - \omega_2)}{m_2 \omega_1 + m_1 \omega_2} \right. \\ &\quad \left. + f_1(q_{P\perp}) \not{q}_{P\perp} \frac{P(\omega_1 + \omega_2)}{m_2 \omega_1 + m_1 \omega_2} \right] \gamma_5. \end{aligned} \quad (\text{B3})$$

According to Eq. (A13) we can further obtain the wave function corresponding to the positive projection:

$$\varphi_{0+}^{++}(q_{P\perp}) = L(N + \frac{P}{M} + \not{q}_{P\perp} Y + \not{q}_{P\perp} \frac{P}{M} Z) \gamma_5, \quad (\text{B4})$$

where

$$\begin{aligned} L &= \frac{M}{2} (f_1 + f_2 \frac{m_1 + m_2}{\omega_1 + \omega_2}), \\ N &= \frac{\omega_1 + \omega_2}{m_1 + m_2}, \\ Y &= \frac{m_2 - m_1}{m_2 \omega_1 + m_1 \omega_2}, \\ Z &= \frac{\omega_1 + \omega_2}{m_2 \omega_1 + m_1 \omega_2}. \end{aligned}$$

The normalization condition reads:

$$\begin{aligned} \int \frac{d\vec{q}}{(2\pi)^3} 4f_1 f_2 M^2 \left\{ \frac{m_1 + m_2}{\omega_1 + \omega_2} + \frac{\omega_1 + \omega_2}{m_1 + m_2} \right. \\ \left. + \frac{2\vec{q}^2 (m_1 \omega_1 + m_2 \omega_2)}{(m_2 \omega_1 + m_1 \omega_2)^2} \right\} = 2M. \end{aligned} \quad (\text{B5})$$

Putting Eq. (B3) into the first two equations of Eq. (A14), we obtain two coupled integral equations about $f_1(q_{P\perp})$ and $f_2(q_{P\perp})$, then by solving them, we obtain $f_1(q_{P\perp})$ and $f_2(q_{P\perp})$, i.e., finally the numerical relativistic wave functions Eq. (B3) with $f_1(q_{P\perp})$ and $f_2(q_{P\perp})$ being given for the corresponding pseudoscalar mesons are obtained. Since the B_c and η_c , B_s etc are pseudoscalar mesons, so the relativistic wave functions of them, which are needed in calculating the weak current matrix elements for the concerned semileptonic decays of B_c , are

obtained in this way. Note that s -quark has a mass $m_s \sim 0.5$ GeV, here we consider it is still “heavy” although people consider it is light one, thus for the same reason we are quite sure that the results about B_s are not so good as those about η_c and B_c etc. The same note for B_s^* is applicable in the next subsection.

2. Equation and solution for heavy vector meson

The relativistic wave function of heavy vector state ($J^P = 1^-$) generally has 8 terms based on P , $q_{P\perp}$, ϵ (polarization vector) and gamma matrices, so the general form for the relativistic Salpeter wave function for 1^- states may be read as [33, 38]:

$$\begin{aligned} \varphi_{1-}^\lambda(q_{P\perp}) = & q_{P\perp} \cdot \epsilon_\perp^\lambda \left[f_1(q_{P\perp}) + f_2(q_{P\perp}) \frac{P}{M} \right. \\ & + f_3(q_{P\perp}) \frac{\not{q}_{P\perp}}{M} + f_4(q_{P\perp}) \frac{P \not{q}_{P\perp}}{M^2} \\ & + f_5(q_{P\perp}) M \not{\epsilon}_\perp^\lambda + f_6(q_{P\perp}) \not{\epsilon}_\perp^\lambda P \\ & + f_7(q_{P\perp}) (\not{q}_{P\perp} \not{\epsilon}_\perp^\lambda - q_{P\perp} \cdot \epsilon_\perp^\lambda) \\ & \left. + f_8(q_{P\perp}) \frac{(P \not{\epsilon}_\perp^\lambda \not{q}_{P\perp} - P q_{P\perp} \cdot \epsilon_\perp^\lambda)}{M} \right] \end{aligned} \quad (B6)$$

where the M is the mass of the vector meson. The equations $\varphi_{0-}^{+-} = \varphi_{0-}^{-+} = 0$ give the following constrains on the components of the wave function:

$$\begin{aligned} f_1(q_{P\perp}) = & \left[f_3(q_{P\perp}) q_{P\perp}^2 + f_5(q_{P\perp}) M^2 \right] \\ & \times \frac{(m_1 m_2 - \omega_1 \omega_2 + q_{P\perp}^2)}{M(m_1 + m_2) q_{P\perp}^2}, \\ f_7(q_{P\perp}) = & \frac{f_5(q_{P\perp}) M(-\omega_1 + \omega_2)}{m_2 \omega_1 + m_1 \omega_2}, \\ f_2(q_{P\perp}) = & \left[-f_4(q_{P\perp}) q_{P\perp}^2 + f_6(q_{P\perp}) M^2 \right] \\ & \times \frac{(m_1 \omega_2 - m_2 \omega_1)}{M(\omega_1 + \omega_2) q_{P\perp}^2}, \\ f_8(q_{P\perp}) = & \frac{f_6(q_{P\perp}) M(\omega_1 \omega_2 - m_1 m_2 - q_{P\perp}^2)}{(m_1 + m_2) q_{P\perp}^2}. \end{aligned} \quad (B7)$$

Putting the constrains into Eq. (B6), one can rewrite the relativistic Salpeter wave function for the states 1^-

as:

$$\begin{aligned} \varphi_{1-}^\lambda(q_{P\perp}) = & q_{P\perp} \cdot \epsilon_\perp^\lambda \left\{ \frac{[f_3(q_{P\perp}) q_{P\perp}^2 + f_5(q_{P\perp}) M^2]}{M(m_1 + m_2)} \right. \\ & \times \frac{(m_1 m_2 - \omega_1 \omega_2 + q_{P\perp}^2)}{q_{P\perp}^2} \\ & + \left[-f_4(q_{P\perp}) q_{P\perp}^2 + f_6(q_{P\perp}) M^2 \right] \\ & \times \frac{(m_1 \omega_2 - m_2 \omega_1)}{M^2(\omega_1 + \omega_2) q_{P\perp}^2} \\ & + f_3(q_{P\perp}) \frac{\not{q}_{P\perp}}{M} + f_4(q_{P\perp}) \frac{P \not{q}_{P\perp}}{M^2} \left. \right\} \\ & + f_5(q_{P\perp}) M \not{\epsilon}_\perp^\lambda + f_6(q_{P\perp}) \not{\epsilon}_\perp^\lambda P \\ & + \frac{f_5(q_{P\perp}) M(-\omega_1 + \omega_2)}{m_2 \omega_1 + m_1 \omega_2} (\not{q}_{P\perp} \not{\epsilon}_\perp^\lambda - q_{P\perp} \cdot \epsilon_\perp^\lambda) \\ & + \frac{f_6(q_{P\perp}) (\omega_1 \omega_2 - m_1 m_2 - q_{P\perp}^2)}{(m_1 + m_2) q_{P\perp}^2} \\ & \times (P \not{\epsilon}_\perp^\lambda \not{q}_{P\perp} - P q_{P\perp} \cdot \epsilon_\perp^\lambda). \end{aligned} \quad (B8)$$

Furthermore, we can obtain the wave function corresponding to the positive projection by Eq. (A13):

$$\begin{aligned} \varphi_{1-}^{++}(q_{P\perp}) = & A \not{\epsilon}_\perp^\lambda + B \not{\epsilon}_\perp^\lambda P + C (\not{q}_{P\perp} \not{\epsilon}_\perp^\lambda - q_{P\perp} \cdot \epsilon_\perp^\lambda) \\ & + D (P \not{\epsilon}_\perp^\lambda \not{q}_{P\perp} - P q_{P\perp} \cdot \epsilon_\perp^\lambda) + q_{P\perp} \cdot \epsilon_\perp^\lambda \\ & \times (E + F P + G \not{q}_{P\perp} + H P \not{q}_{P\perp}), \end{aligned} \quad (B9)$$

where

$$\begin{aligned} A = & \frac{1}{2} M (f_5 - f_6 \frac{\omega_1 + \omega_2}{m_1 + m_2}), \\ B = & \frac{1}{2} (f_6 - f_5 \frac{m_1 + m_2}{\omega_1 + \omega_2}), \\ C = & \frac{1}{2} M \frac{\omega_2 - \omega_1}{m_2 \omega_1 + m_1 \omega_2} (f_5 - f_6 \frac{\omega_1 + \omega_2}{m_1 + m_2}), \\ D = & \frac{1}{2} \frac{\omega_1 + \omega_2}{\omega_1 \omega_2 + m_1 m_2 + \bar{q}^2} (f_5 - f_6 \frac{\omega_1 + \omega_2}{m_1 + m_2}), \\ E = & \frac{1}{2} \frac{m_1 + m_2}{M(\omega_1 \omega_2 + m_1 m_2 - \bar{q}^2)} [M^2 (f_5 \\ & - f_6 \frac{m_1 + m_2}{\omega_1 + \omega_2}) - \bar{q}^2 (f_3 + f_4 \frac{m_1 + m_2}{\omega_1 + \omega_2})], \\ F = & \frac{1}{2} \frac{\omega_1 - \omega_2}{M^2(\omega_1 \omega_2 + m_1 m_2 - \bar{q}^2)} [M^2 (f_5 \\ & - f_6 \frac{m_1 + m_2}{\omega_1 + \omega_2}) - \bar{q}^2 (f_3 + f_4 \frac{m_1 + m_2}{\omega_1 + \omega_2})], \\ G = & \frac{1}{2} \left[\frac{1}{M} (f_3 + f_4 \frac{m_1 + m_2}{\omega_1 + \omega_2}) - \frac{2 f_6 M}{m_2 \omega_1 + m_1 \omega_2} \right], \\ H = & \frac{1}{2} \frac{1}{M^2} \left[(f_3 \frac{\omega_1 + \omega_2}{m_1 + m_2} + f_4) - 2 f_5 M^2 \right. \\ & \left. \times \frac{\omega_1 + \omega_2}{(m_1 + m_2)(\omega_1 \omega_2 + m_1 m_2 + \bar{q}^2)} \right]. \end{aligned}$$

The normalization condition now is read as below:

$$\int \frac{d\vec{q}}{(2\pi)^3} \frac{16\omega_1\omega_2}{3} \left\{ 3f_5f_6 \frac{M^2}{m_2\omega_1 + m_1\omega_2} + \frac{\omega_1\omega_2 - m_1m_2 + \vec{q}^2}{(m_1 + m_2)(\omega_1 + \omega_2)} \times \left[f_4f_5 - f_3(f_4 \frac{\vec{q}^2}{M^2} + f_6) \right] \right\} = 2M. \quad (\text{B10})$$

From the first two equations of Eq. (A14) and in terms of straightforward calculation, one may obtain four coupled integral equations about $f_3(q_{P_\perp})$, $f_4(q_{P_\perp})$, $f_5(q_{P_\perp})$ and $f_6(q_{P_\perp})$. By solving them one may obtain the numerical results for the mass M and the relativistic wave function Eq. (B8) with $f_3(q_{P_\perp})$, $f_4(q_{P_\perp})$, $f_5(q_{P_\perp})$ and $f_6(q_{P_\perp})$ being given. Since the J/ψ and B_s^* etc are vector mesons, so for the concerned semileptonic decays of B_c , all the relativistic wave functions, which are needed in calculating the weak current matrix elements, are obtained in the present way.

3. The weak current matrix elements and form factors

For $B_c^+ \rightarrow P\ell^+\bar{\nu}_\ell$ (here we take $P = B_s$ for example), the hadron matrix element Eq. (7) based on the positive energy wave function of pseudoscalar meson Eq. (B4) can be written as:

$$\begin{aligned} & \langle B_s(P') | J_\mu | B_c^+(P) \rangle \\ &= \int \frac{d^3\vec{q}}{(2\pi)^3} 4L' L \left(\frac{P_\mu}{M} s_1 + \frac{P'_\mu}{M'} s_2 + q_{P_\perp \mu} s_3 \right) \\ &= S_1 \frac{P_\mu}{M} + S_2 \frac{P'_\mu}{M'} + S_3 (P'_\mu - \frac{E_f}{M} P_\mu) \\ &= P_\mu \left(\frac{S_1}{M} - \frac{E_f}{M} S_3 \right) + P'_\mu \left(\frac{S_2}{M'} + S_3 \right) \\ &= P_\mu (f_+ + f_-) + P'_\mu (f_+ - f_-) \\ &= f_+(P + P')_\mu + f_-(P - P')_\mu, \end{aligned} \quad (\text{B11})$$

where E_f is the energy of the meson in final state, and

$$\begin{aligned} L' &= \frac{M'}{2} (f'_1 + f'_2 \frac{m'_1 + m'_2}{\omega'_1 + \omega'_2}), \\ N' &= \frac{\omega'_1 + \omega'_2}{m'_1 + m'_2}, \\ Y' &= \frac{m'_2 - m'_1}{m'_2\omega'_1 + m'_1\omega'_2}, \\ Z' &= \frac{\omega'_1 + \omega'_2}{m'_2\omega'_1 + m'_1\omega'_2}; \end{aligned}$$

$$\begin{aligned} s_1 &= N'N + \frac{Y}{M'} \vec{r} \cdot \vec{q} - Y' \alpha'_2 E_f \\ &\quad + Y'Y(\vec{q}^2 + \alpha'_2 \vec{r} \cdot \vec{q}) + \frac{Z'Z}{M'} \alpha'_2 E_f \vec{r} \cdot \vec{q} \\ &\quad - \frac{Z'N}{M'} (\vec{r} \cdot \vec{q} + \alpha'_2 \vec{r}^2 + \alpha'_2 E_f^2), \\ s_2 &= 1 + Y' \alpha'_2 M' + Z'N \alpha'_2 E_f + Z'Z \vec{q}^2, \\ s_3 &= N'Z + \frac{Y}{M'} E_f + Y' + \frac{Z'N}{M'} E_f \\ &\quad - \frac{Z'Z}{M'} (2\vec{r} \cdot \vec{q} + \alpha'_2 \vec{r}^2); \end{aligned}$$

$$\begin{aligned} S_1 &= \int \frac{d^3\vec{q}}{(2\pi)^3} 4L' L s_1, \\ S_2 &= \int \frac{d^3\vec{q}}{(2\pi)^3} 4L' L s_2, \\ S_3 &= \frac{1}{|\vec{r}|} \int \frac{d^3\vec{q}}{(2\pi)^3} |\vec{q}| \cos \theta 4L' L s_3; \end{aligned}$$

Then the form factors f_+ and f_- in Eq. (8) are defined as:

$$\begin{aligned} f_+ &= \frac{1}{2} \left(\frac{S_1}{M} + \frac{S_2}{M'} + \frac{M - E_f}{M} S_3 \right), \\ f_- &= \frac{1}{2} \left(\frac{S_1}{M} - \frac{S_2}{M'} - \frac{M + E_f}{M} S_3 \right). \end{aligned} \quad (\text{B12})$$

For $B_c^+ \rightarrow V\ell^+\bar{\nu}_\ell$ (here we take $V = B_s^*$ for example), the hadron matrix element Eq. (7) based on the positive energy wave function of pseudoscalar meson Eq. (B4)

and vector meson Eq. (B9) can be written as:

$$\begin{aligned}
& \langle B_s^*(P', \epsilon) | J_\mu | B_c^+(P) \rangle \\
&= \int \frac{d^3 \vec{q}}{(2\pi)^3} 4L \left\{ \epsilon'_{\perp\mu} t_1 + P_\mu \left[(q_{P\perp} \cdot \epsilon'_\perp) t_2 + (P \cdot \epsilon'_\perp) t'_2 \right] \right. \\
&+ P'_\mu \left[(q_{P\perp} \cdot \epsilon'_\perp) t_3 + (P \cdot \epsilon'_\perp) t'_3 \right] \\
&+ q_{P\perp\mu} \left[(q_{P\perp} \cdot \epsilon'_\perp) t_4 + (P \cdot \epsilon'_\perp) t'_4 \right] \\
&- i \varepsilon_{\mu\nu\rho\sigma} \left[\frac{A'Y}{M} \epsilon'^{\lambda\nu}_\perp q_{P\perp}^\rho P^\sigma - \frac{B'N}{M} P'^\nu \epsilon'^{\lambda\rho}_\perp P^\sigma \right. \\
&- B'Z P'^\nu \epsilon'^{\lambda\rho}_\perp q_{P\perp}^\sigma - \frac{C'N}{M} \epsilon'^{\lambda\nu}_\perp q_{P\perp}^\rho P^\sigma \\
&- \frac{C'N}{M} \alpha'_2 \epsilon'^{\lambda\nu}_\perp P'^\rho P^\sigma - C'Z \alpha'_2 \epsilon'^{\lambda\nu}_\perp P'^\rho q_{P\perp}^\sigma \\
&+ \frac{C'Z}{M} \alpha'_2 E_f \epsilon'^{\lambda\nu}_\perp P^\rho q_{P\perp}^\sigma + D'q_{P\perp}^\nu \epsilon'^{\lambda\rho}_\perp P'^\sigma \\
&- \frac{D'}{M} \alpha'_2 E_f P^\nu \epsilon'^{\lambda\rho}_\perp P'^\sigma + \frac{D'Y}{M} \vec{q}^2 \epsilon'^{\lambda\nu}_\perp P'^\rho P^\sigma \\
&- \frac{D'Y}{M} \alpha'_2 M'^2 \epsilon'^{\lambda\nu}_\perp q_{P\perp}^\rho P^\sigma - D'Y \alpha'_2 E_f \epsilon'^{\lambda\nu}_\perp P'^\rho q_{P\perp}^\sigma \\
&- (q_{P\perp} \cdot \epsilon'_\perp) \left(\frac{D'Y}{M} P'^\nu q_{P\perp}^\rho P^\sigma - \frac{F'Y}{M} P'^\nu q_{P\perp}^\rho P^\sigma \right. \\
&- \frac{G'Y}{M} \alpha'_2 P'^\nu q_{P\perp}^\rho P^\sigma + \frac{H'N}{M} q_{P\perp}^\nu P'^\rho P^\sigma \\
&- \left. \left. \frac{H'Z}{M} \alpha'_2 E_f P^\nu P'^\rho q_{P\perp}^\sigma \right) \right] \Big\} \\
&= (T_1 + T_{43}) \epsilon'^\lambda_{\perp\mu} + (T_2 + T'_2 + T_{41} + T'_{41}) (P \cdot \epsilon'_\perp) P_\mu \\
&+ (T_3 + T'_3 + T_{42} + T'_{42}) (P \cdot \epsilon'_\perp) P'_\mu \\
&+ i \varepsilon_{\mu\nu\rho\sigma} \epsilon'^{\lambda\nu}_\perp P'^\rho P^\sigma (M_1 - M_2 + M_3 + M_4 \\
&- M_5 + M_6 + M_7 + M_8 - M_9 - M_{10} - M_{11} \\
&+ M_{12} - M_{13} - V_1 + V_2 + V_3 + V_4) \\
&= f \epsilon'^\lambda_{\perp\mu} + a_+ (P \cdot \epsilon'_\perp) (P + P')_\mu \\
&+ a_- (P \cdot \epsilon'_\perp) (P - P')_\mu \\
&+ i g \varepsilon_{\mu\nu\rho\sigma} \epsilon'^{\lambda\nu}_\perp (P + P')^\rho (P - P')^\sigma, \quad (B13)
\end{aligned}$$

where the definition of A' , B' , C' , D' , E' , F' , G' and H' is the same as Eq. (B9) but for final meson, and

$$\begin{aligned}
t_1 &= A' - B'N E_f + B'Z \vec{r} \cdot \vec{q} \\
&- C'Z (\vec{q}^2 + \alpha'_2 \vec{r} \cdot \vec{q}) + D' (\alpha'_2 \vec{r}^2 + \vec{r} \cdot \vec{q}) \\
&- D'Y E_f (\vec{q}^2 + \alpha'_2 \vec{r} \cdot \vec{q}), \\
t_2 &= -\frac{A'Y}{M} + \frac{D'Y}{M} (\alpha'_2 M'^2 - \vec{r} \cdot \vec{q}) - \frac{E'N}{M} \\
&+ \frac{F'Y}{M} \vec{r} \cdot \vec{q} + \frac{G'Y}{M} (\vec{q}^2 + \alpha'_2 \vec{r} \cdot \vec{q}) \\
&- \frac{H'N}{M} (\alpha'_2 M'^2 - \vec{r} \cdot \vec{q}) + \frac{C'Z}{M} \alpha'_2 E_f \\
&- \frac{G'}{M} \alpha'_2 E_f + \frac{H'Z}{M} \alpha'_2 E_f \vec{r} \cdot \vec{q},
\end{aligned}$$

$$\begin{aligned}
t'_2 &= \frac{D'Y}{M} \frac{\alpha'_2 E_f}{M} \vec{r} \cdot \vec{q} - \frac{F'Y}{M} \frac{\alpha'_2 E_f}{M} \vec{r} \cdot \vec{q} \\
&- \frac{G'Y}{M} \frac{\alpha'_2 E_f}{M} (\vec{q}^2 + \alpha'_2 \vec{r} \cdot \vec{q}) \\
&+ \frac{H'N}{M} \frac{\alpha'_2 E_f}{M} (\alpha'_2 M'^2 - \vec{r} \cdot \vec{q}) \\
&- \frac{H'Z}{M} \frac{\alpha'^2_2 E_f^2}{M} \vec{r} \cdot \vec{q} + \frac{E'N}{M} \frac{\alpha'_2 E_f}{M} \\
&+ \frac{C'N}{M} \frac{\alpha'_2 E_f}{M} + \frac{G'}{M} \frac{\alpha'^2_2 E_f^2}{M}, \\
t_3 &= B'Z - D'Y \alpha'_2 E_f + F' + H'Z \vec{q}^2 \\
&- C'Z \alpha'_2 + G' \alpha'_2 + H'N \alpha'_2 E_f, \\
t'_3 &= \frac{B'N}{M} + \frac{D'Y}{M} \vec{q}^2 - \frac{F'}{M} \alpha'_2 E_f - \frac{C'N}{M} \alpha'_2 \\
&- \frac{H'Z}{M} \alpha'_2 E_f \vec{q}^2 - \frac{G'}{M} \alpha'^2_2 E_f - \frac{H'N}{M} \alpha'^2_2 E_f^2, \\
t_4 &= -D'Y E_f - E'Z + F'Y E_f - C'Z \\
&+ G' + H'Z \alpha'_2 \vec{r}^2 + H'N E_f, \\
t'_4 &= \frac{A'Y}{M} + \frac{D'Y}{M} \alpha'_2 \vec{r}^2 + \frac{E'Z}{M} \alpha'_2 E_f - \frac{C'N}{M} \\
&- \frac{F'Y}{M} \alpha'_2 E_f^2 - \frac{H'Z}{M} \alpha'^2_2 E_f \vec{r}^2 \\
&- \frac{G'}{M} \alpha'_2 E_f - \frac{H'N}{M} \alpha'_2 E_f^2;
\end{aligned}$$

$$\begin{aligned}
T_1 &= \int \frac{d^3 \vec{q}}{(2\pi)^3} 4A t_1, \\
T_2 &= -\frac{1}{|\vec{r}|} \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{E_f}{M} |\vec{q}| \cos \theta 4A t_2, \\
T'_2 &= \int \frac{d^3 \vec{q}}{(2\pi)^3} 4A t'_2, \\
T_3 &= -\frac{1}{|\vec{r}|} \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{E_f}{M} |\vec{q}| \cos \theta 4A t_3, \\
T'_3 &= \int \frac{d^3 \vec{q}}{(2\pi)^3} 4A t'_3, \\
T_{41} &= \frac{1}{2M^2 |\vec{r}|^2} \int \frac{d^3 \vec{q}}{(2\pi)^3} |\vec{q}|^2 \\
&\times [(M'^2 + 2E_f^2) \cos^2 \theta - M'^2] 4A t_4, \\
T'_{41} &= -\frac{1}{|\vec{r}|} \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{E_f}{M} |\vec{q}| \cos \theta 4A t'_4, \\
T_{42} &= -\frac{E_f}{2M |\vec{r}|^2} \int \frac{d^3 \vec{q}}{(2\pi)^3} |\vec{q}|^2 (3 \cos^2 \theta - 1) 4A t_4, \\
T'_{42} &= \frac{1}{|\vec{r}|} \int \frac{d^3 \vec{q}}{(2\pi)^3} |\vec{q}| \cos \theta 4A t'_4, \\
T_{43} &= \frac{1}{2} \int \frac{d^3 \vec{q}}{(2\pi)^3} |\vec{q}|^2 (\cos^2 \theta - 1) 4A t_4;
\end{aligned}$$

$$\begin{aligned}
M_1 &= -\frac{1}{|\vec{r}|} \int \frac{d^3\vec{q}}{(2\pi)^3} |\vec{q}| \cos\theta 4L \frac{A'Y}{M}, \\
M_2 &= -\frac{1}{|\vec{r}|} \int \frac{d^3\vec{q}}{(2\pi)^3} E_f |\vec{q}| \cos\theta 4L \frac{B'Z}{M}, \\
M_3 &= \frac{1}{|\vec{r}|} \int \frac{d^3\vec{q}}{(2\pi)^3} |\vec{q}| \cos\theta 4L \frac{C'N}{M}, \\
M_4 &= -\frac{1}{|\vec{r}|} \int \frac{d^3\vec{q}}{(2\pi)^3} \alpha'_2 E_f |\vec{q}| \cos\theta 4L \frac{C'Z}{M}, \\
M_5 &= -\frac{1}{|\vec{r}|} \int \frac{d^3\vec{q}}{(2\pi)^3} \alpha'_2 E_f |\vec{q}| \cos\theta 4L \frac{C'Z}{M}, \\
M_6 &= \frac{1}{|\vec{r}|} \int \frac{d^3\vec{q}}{(2\pi)^3} E_f |\vec{q}| \cos\theta 4L \frac{D'}{M}, \\
M_7 &= \frac{1}{|\vec{r}|} \int \frac{d^3\vec{q}}{(2\pi)^3} \alpha'_2 M'^2 |\vec{q}| \cos\theta 4L \frac{D'Y}{M}, \\
M_8 &= -\frac{1}{|\vec{r}|} \int \frac{d^3\vec{q}}{(2\pi)^3} \alpha'_2 E_f^2 |\vec{q}| \cos\theta 4L \frac{D'Y}{M}, \\
M_9 &= \frac{1}{2} \int \frac{d^3\vec{q}}{(2\pi)^3} |\vec{q}|^2 (\cos^2\theta - 1) 4L \frac{D'Y}{M}, \\
M_{10} &= -\frac{1}{2} \int \frac{d^3\vec{q}}{(2\pi)^3} |\vec{q}|^2 (\cos^2\theta - 1) 4L \frac{F'Y}{M}, \\
M_{11} &= -\frac{1}{2} \int \frac{d^3\vec{q}}{(2\pi)^3} \alpha'_2 |\vec{q}|^2 (\cos^2\theta - 1) 4L \frac{G'Y}{M}, \\
M_{12} &= \frac{1}{2} \int \frac{d^3\vec{q}}{(2\pi)^3} |\vec{q}|^2 (\cos^2\theta - 1) 4L \frac{H'N}{M}, \\
M_{13} &= -\frac{1}{2} \int \frac{d^3\vec{q}}{(2\pi)^3} \alpha'_2 E_f |\vec{q}|^2 (\cos^2\theta - 1) 4L \frac{H'Z}{M}; \\
V_1 &= \int \frac{d^3\vec{q}}{(2\pi)^3} 4L \frac{B'N}{M}, \\
V_2 &= \int \frac{d^3\vec{q}}{(2\pi)^3} \alpha'_2 4L \frac{C'N}{M}, \\
V_3 &= \int \frac{d^3\vec{q}}{(2\pi)^3} \alpha'_2 E_f 4L \frac{D'}{M}, \\
V_4 &= - \int \frac{d^3\vec{q}}{(2\pi)^3} |\vec{q}|^2 4L \frac{D'Y}{M};
\end{aligned}$$

Then the form factors f , a_+ , a_- and g in Eq. (9) and (10) are defined as:

$$\begin{aligned}
f &= T_1 + T_{43}, \\
a_+ &= \frac{1}{2}(T_2 + T'_2 + T_{41} + T'_{41} \\
&\quad + T_3 + T'_3 + T_{42} + T'_{42}), \\
a_- &= \frac{1}{2}(T_2 + T'_2 + T_{41} + T'_{41} \\
&\quad - T_3 - T'_3 - T_{42} - T'_{42}), \\
g &= \frac{1}{2}(M_1 - M_2 + M_3 + M_4 - M_5 \\
&\quad + M_6 + M_7 + M_8 - M_9 - M_{10} \\
&\quad - M_{11} + M_{12} - M_{13} - V_1 \\
&\quad + V_2 + V_3 + V_4). \tag{B14}
\end{aligned}$$

4. The Parameters in QCD inspired BS Equation

When solving the equations, we have to fix the BS (instantaneous) kernel. Considering the successes of Cornell

potential model on heavy quarkonia[39], we would like to refer the BS kernel to the model. Moreover, the color factor for the relevant BS equation may be factorized out straightforwardly, thus we leave the factor aside, and focus on the rest factors of the formulation for the kernel. They are a linear scalar interaction $V_s(r) = \lambda r$ for ‘color-confinement’, a vector interaction $V_v(r) = -\frac{4}{3} \frac{\alpha_s(r)}{r}$ for one-gluon exchange, i.e.:

$$\begin{aligned}
I(r) &= V_s(r) + V_0 + \gamma_0 \otimes \gamma^0 V_v(r) \\
&= \lambda r + V_0 - \gamma_0 \otimes \gamma^0 \frac{4}{3} \frac{\alpha_s(r)}{r}, \tag{B15}
\end{aligned}$$

where λ is the so-called ‘string constant’, $\alpha_s(r)$ is the running coupling constant, and a constant V_0 , which, as a ‘zero-point’, is added.

The kernel in momentum space reads:

$$I(\vec{q}) = V_s(\vec{q}) + \gamma_0 \otimes \gamma^0 V_v(\vec{q}), \tag{B16}$$

where

$$V_s(\vec{q}) = -\left(\frac{\lambda}{\alpha} + V_0\right) \delta^3(\vec{q}) + \frac{\lambda}{\pi^2} \frac{1}{(\vec{q}^2 + \alpha^2)^2},$$

$$V_v(\vec{q}) = -\frac{2}{3\pi^2} \frac{\alpha_s(\vec{q})}{(\vec{q}^2 + \alpha^2)},$$

and

$$\alpha_s(\vec{q}) = \frac{12\pi}{27} \frac{1}{\log(a + \frac{\vec{q}^2}{\Lambda_{QCD}^2})}.$$

In order to avoid the Coulomb-like infrared divergence, usually a factor $e^{-\alpha r}$ as below:

$$\begin{aligned}
V_s(r) &= \frac{\lambda}{\alpha} (1 - e^{-\alpha r}), \\
V_v(r) &= -\frac{4}{3} \frac{\alpha_s(r)}{r} e^{-\alpha r}. \tag{B17}
\end{aligned}$$

is introduced.

The parameters λ , α , a and Λ_{QCD} characterizing the potential are fixed by fitting the mass spectrum of heavy quarkonium [33]. The fitted values are $a = e = 2.7183$, $\alpha = 0.06$ GeV, $\lambda = 0.21$ GeV², $\Lambda_{QCD} = 0.27$ GeV. The parameter V_0 varies as the constituents and the quantum number of the concerned meson being varying. In this work, the relevant values are $V_0 = -0.314$ GeV for $\bar{c}c(0^{++})$, $V_0 = -0.176$ GeV for $\bar{c}c(1^{--})$, $V_0 = -0.205$ GeV for $\bar{b}s(0^-)$, $V_0 = -0.13$ GeV for $\bar{b}s(1^-)$ and $V_0 = -0.185$ GeV for $\bar{b}c(0^-)$. The constituent quark masses are parameters too, and they are fixed by fitting the meson spectrum: $m_b = 4.96$ GeV, $m_c = 1.62$ GeV, $m_s = 0.5$ GeV. With these parameters, we obtain the mass spectrum and the relevant wave functions by solving the precise equations obtained in previous subsection.

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- [1] C. H. Chang and Y. Q. Chen, Phys. Rev. D **48**, 4086 (1993); C. H. Chang, Y. Q. Chen, G. P. Han and H. T. Jiang, Phys. Lett. B **364**, 78 (1995); C. H. Chang, Y. Q. Chen and R. J. Oakes, Phys. Rev. D **54**, 4344 (1996); C. H. Chang and Y. Q. Chen, Phys. Rev. D **46**, 3845 (1992). C. H. Chang and X. G. Wu, Eur. Phys. J. C **38** 267, (2004) [arXiv:hep-ph/0309121].
- [2] N. Brambilla *et al.*, Eur. Phys. J. C **71**, 1534 (2011) [arXiv:1010.5827] and references therein.
- [3] C. H. Chang, Int. J. Mod. Phys. A **21**, 777 (2006) [arXiv:hep-ph/0509211].
- [4] CDF Collaboration, F. Abe *et al.*, Phys. Rev. Lett. **81**, 2432 (1998); Phys. Rev. D **58**, 112004 (1998).
- [5] CDF Collaboration, T. Aaltonen *et al.*, Phys. Rev. Lett. **96**, 182002 (2006).
- [6] CDF Collaboration, T. Aaltonen *et al.*, Phys. Rev. Lett. **100**, 182002 (2008).
- [7] D0 Collaboration, V. M. Abazov *et al.*, Phys. Rev. Lett. **101**, 012001 (2008); Phys. Rev. Lett. **102**, 092001 (2009).
- [8] LHCb Collaboration, R. Asij *et al.*, Phys. Rev. **D87**, 112012 (2013); Phys. Rev. **D87**, 071103(R) (2013).
- [9] C. H. Chang and Y. Q. Chen, Phys. Rev. D **49**, 3399 (1994).
- [10] N. Isgur, D. Scora, B. Grinstein and M. Wise, Phys. Rev. D **39**, 799 (1989).
- [11] V. V. Kiselev, A. E. Kovalsky and A. K. Likhoded, Nucl. Phys. B **585**, 353 (2000) [arXiv:hep-ph/0002127]; V. V. Kiselev, A. K. Likhoded and A. I. Onishchenko, Nucl. Phys. B **569**, 473 (2000); V. V. Kiselev, arXiv:hep-ph/0211021; V. V. Kiselev, arXiv:hep-ph/0308214.
- [12] C. H. Chang, Y. Q. Chen, G. L. Wang and H. S. Zong, Phys. Rev. D **65**, 014017 (2002) [arXiv:hep-ph/0103036]; Commun. Theor. Phys. **35**, 395 (2001).
- [13] C. H. Chang, S. L. Chen, T. F. Feng and X. Q. Li, Commun. Theor. Phys. **35**, 51 (2001); Phys. Rev. D **64**, 014003 (2001).
- [14] P. Colangelo and F. De Fazio, Phys. Rev. D **61**, 034012 (2000) [arXiv:hep-ph/9909423].
- [15] A. Abd El-Hady, J. H. Muñoz and J. P. Vary, Phys. Rev. D **62**, 014019 (2000) [arXiv:hep-ph/9909406].
- [16] A. Y. Anisimov, I. M. Narodetskii, C. Semay and B. Silvestre-Brac, Phys. Lett. B **452**, 129 (1999) [arXiv:hep-ph/9812514]; A. Y. Anisimov, P. Y. Kulikov, I. M. Narodetskii and K. A. Ter-Martirosyan, Phys. Atom. Nucl. **62**, 1739 (1999) [arXiv:hep-ph/9809249].
- [17] D. Ebert, R. N. Faustov and V. O. Galkin, Mod. Phys. Lett. A **17**, 803 (2002) [arXiv:hep-ph/0204167]; Eur. Phys. J. C **32**, 29 (2003) [arXiv:hep-ph/0308149]; Phys. Rev. D **68**, 094020 (2003) [arXiv:hep-ph/0306306].
- [18] M. A. Ivanov, J. G. Körner and P. Santorelli, Phys. Rev. D **63**, 074010 (2001); Phys. Rev. D **71**, 094006 (2005) [arXiv:hep-ph/0501051]; Phys. Rev. D **73**, 054024 (2006) [arXiv:hep-ph/0602050].
- [19] M. A. Sanchis-Lozano, Nucl. Phys. B **440**, 251 (1995).
- [20] M. A. Nobes and R. M. Woloshyn, J. Phys. G **26**, 1079 (2000) [arXiv:hep-ph/0005056].
- [21] G. R. Lu, Y. D. Yang and H. B. Li, Phys. Lett. B **341**, 391 (1995).
- [22] M. Lusignoli and M. Masetti, Z. Phys. C **51**, 549 (1991).
- [23] J. F. Liu and K. T. Chao, Phys. Rev. D **56**, 4133 (1997).
- [24] D. S. Du and Z. Wang, Phys. Rev. D **39**, 1342 (1989).
- [25] E. Hernández, J. Nieves and J. M. Verde-Velasco, Phys. Rev. D **74**, 074008 (2006) [arXiv:hep-ph/0607150].
- [26] Y. M. Wang, C. D. Lü, Phys. Rev. D **77**, 054003 (2008).
- [27] H.-M. Choi and C.-R. Ji, Phys. Rev. D **80**, 114003 (2009) [arXiv:0909.5028].
- [28] W.-F. Wang, Y.-Y. Fan, Z.-J. Xiao, Chin Phys. C **37**, 093102 (2013); Z.-J. Xiao, X. Liu, Chin Sci Bull **59**, 3748 (2014).
- [29] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
- [30] E. E. Salpeter, Phys. Rev. **87**, 328 (1952).
- [31] S. Mandelstam, Proc. R. Soc. London **233**, 248 (1955).
- [32] C. H. Chang and G. L. Wang, Sci. China G **53** 2005-2018, (2010), [arXiv:1003.3827]; C. H. Chang and J. K. Chen, Commun. Theor. Phys. **44**, 646 (2005) [arXiv:nucl-th/0409077]; C. H. Chang, J. K. Chen, X.-Q. Li and G. L. Wang, Commun. Theor. Phys. **43**, 113 (2005) [arXiv:hep-ph/0406050].
- [33] C. H. Chang, J. K. Chen and G. L. Wang, Commun. Theor. Phys. **46** 467, (2006) [arXiv:hep-th/0312350].
- [34] M. K. Gaillard and B. W. Lee, Phys. Rev. Lett. **33**, 108 (1974); G. Altarelli and L. Maiani, Phys. Lett. B **52**, 351 (1974); F. G. Gilman and M. B. Wise, Phys. Rev. D **20**, 2392 (1979).
- [35] K. A. Olive, et al (PDG), Chinese Physics C, **38** 090001 (2014).
- [36] P. Ball and R. Zwicky, Phys. Rev. D **71**, 014029 (2005) [arXiv:hep-ph/0412079].
- [37] C. S. Kim and G. L. Wang, Phys. Lett. B **584**, 285 (2004) [arXiv:hep-ph/0309162].
- [38] G. L. Wang, Phys. Lett. B **633**, 492 (2006) [arXiv:math-ph/0512009].
- [39] E. Eichten, K. Gottfried, T. Kinoshita, *et al.*, Phys. Rev. Lett. **34**, 369 (1973).